

# 2-Degenerate Bertrand curves in Minkowski spacetime

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## Abstract

In this paper we define a new type of 2-degenerate Cartan curves in Minkowski spacetime ( $R_1^4$ ). We prove that this type of curves contain only the polynomial functions as its components whose third derivative vanish completely. No curve with acceleration zero in  $R_1^4$  is a 2-degenerate Cartan curve, therefore we show that the type of curves that we search for must contain polynomials of degree two among its components.

## 1 Introduction

H. Matsuda and S. Yorozu [7] introduced a new type of curves called special Frenet curves and proved that a special Frenet curve in  $R^n$  is not a Bertrand curve if  $n \geq 4$ . They also improved an idea of generalized Bertrand curve in  $R^4$ . A. Ferrandez, A. Gimenez, P. Lucas [4] introduced the notion of  $s$ -degenerate curves in Lorentzian space forms. They obtained a reference along an  $s$ -degenerate curve in an  $n$ -dimensional Lorentzian space with the minimum number of curvatures. That reference generalizes the reference of Bonnor for null curves in Minkowski spacetime and it would be called the Cartan frame of the curve. The associated curvature functions are called the Cartan curvatures of the curve. They characterized the  $s$ -degenerate helices ( i.e,  $s$ -degenerate curves with constant Cartan curvatures ) in  $n$ -dimensional Lorentzian space forms and they obtained a complete classification of them in dimension four.

Let  $C$  be an  $s$ -degenerate Cartan curve in  $R_1^n$ . We call  $W_j$  the spacelike Cartan  $j$ -normal vector along  $C$ , and the spacelike Cartan  $j$ -normal line of  $C$  at  $c(s)$  is a line generated by  $W_j(s)$  through  $c(s)$  ( $j = 1, 2, \dots, n - 2$ ). The spacelike Cartan  $(j, k)$ -normal plane of  $C$  at  $c(s)$  is a plane spanned by  $W_j(s)$  and  $W_k(s)$  through  $c(s)$  ( $j, k = 1, 2, \dots, n - 2; j \neq k$ ). In this paper we characterize 2-degenerate (1,2)-Bertrand curves in Minkowski spacetime.

## 2 Preliminaries

Let  $E$  be a real vector space with a symmetric bilinear mapping  $g : E \times E \rightarrow R$ . We say that  $g$  is degenerate on  $E$  if there exist a vector  $\varepsilon \neq 0$  in  $E$  such that

$$g(\varepsilon, \nu) = 0 \quad \text{for all } \nu \in E,$$

otherwise,  $g$  is said to be non-degenerate. The radical (also called the null space) of  $E$ , with respect to  $g$ , is the subspace  $\text{Rad}(E)$  of  $E$  defined by

$$\text{Rad}(E) = \{\varepsilon \in E \text{ such that } g(\varepsilon, \nu) = 0, \quad \nu \in E\}.$$

For simplicity, we will use  $\langle, \rangle$  instead of  $g$ . A vector  $\nu$  is said to be timelike, lightlike or spacelike provided that  $g(\nu, \nu) < 0$ ,  $g(\nu, \nu) = 0$  (and  $\nu \neq 0$ ), or  $g(\nu, \nu) > 0$  respectively. The vector  $\nu = 0$  is said to be spacelike. A unit vector is a vector  $u$  such that  $g(u, u) = \mp 1$ . Two vectors  $u$  and  $v$  are said to be orthogonal, written  $u \perp v$ , if  $g(u, v) = 0$ .

Let  $(M_1^n, \nabla)$  be an oriented Lorentzian manifold and let  $C : I \rightarrow M_1^n$  be a differentiable curve in  $M_1^n$ . For any vector field  $V$  along  $C$ , Let  $V'$  be the covariant derivative of  $V$  along  $C$ . Write

$$E_i(t) = \text{span} \left\{ c'(t), c''(t), \dots, c^{(i)}(t) \right\},$$

where  $t \in I$  and  $i = 1, 2, \dots, n$ . Let  $d$  be the number defined by

$$d = \max \{i : \dim E_i(t) = i \text{ for all } t\}.$$

With the above notation, the curve  $C : I \rightarrow M_1^n$  is said to be an  $s$ -degenerate (or  $s$ -lightlike) curve if for all  $1 \leq i \leq d$ ,  $\dim \text{Rad}(E_i(t))$  is constant for all  $t$ , and there exist  $s$ ,  $0 \leq s \leq d$ , such that  $\text{Rad}(E_s) \neq \{0\}$  and  $\text{Rad}(E_j) = \{0\}$  for all  $j < s$ . Note that 1-degenerate curves are precisely the null (or lightlike) curves. In this paper we will focus on 2-degenerate curves ( $s = 2$ ) in Minkowski spacetime. Notice that they must be spacelike curves

A *spacetime* is a connected time-oriented four dimensional Lorentz manifold. A *Minkowski spacetime*  $M$  is a spacetime that is isometric to Minkowski 4-space  $R_1^4$  [9]. So  $R_1^4$  is a 4-dimensional Lorentz manifold furnished with the metric  $\langle, \rangle$  defined as follows

$$\langle x, y \rangle = -x^0 y^0 + x^1 x^1 + x^2 x^2 + x^3 x^3$$

for all vectors  $x, y \in R_1^4$ ;  $x = (x^0, x^1, x^2, x^3)$ ,  $y = (y^0, y^1, y^2, y^3)$ ,  $x^i, y^i \in R$ ,  $0 \leq i \leq 3$ .

Let  $C$  be a 2-degenerate Cartan curve in  $R_1^4$ . Then the Cartan equations are in the following form [4].

$$\begin{aligned}
c' &= W_1, \\
W_1' &= L, \\
L' &= k_1 W_2, \\
W_2' &= -k_2 L + k_1 N, \\
N' &= W_1 - k_2 W_2.
\end{aligned}$$

where  $L, N$  are null,  $\langle L, N \rangle = -1$ ,  $\{L, N\}$  and  $\{W_1, W_2\}$  are orthogonal,  $\{W_1, W_2\}$  is orthonormal.  $\{L, N, W_1, W_2\}$  is positively oriented. We assume the set  $\{c', c'', c''', c^{(4)}\}$  has the same orientation with the set  $\{L, N, W_1, W_2\}$ , so we get  $k_1 < 0$ .

Let  $(C, \overline{C})$  be a pair of framed null Cartan curves in  $R_1^4$ , with pseudo-arc parameters  $s$  and  $\overline{s}$ , respectively. This pair is said to be a null Bertrand pair if their spacelike vectors  $W_1$  and  $\overline{W}_1$  are linearly dependent. The curve  $\overline{C}$  is called a Bertrand mate of  $C$  and vice versa. A framed null curve is said to be a null Bertrand curve if it admits a Bertrand mate [3]. To be precise, a null Cartan curve  $C$  in  $R_1^4$  ( $c : I \rightarrow R_1^4$ ) is called a Bertrand curve if there exist a null Cartan curve  $\overline{C}$  ( $\overline{c} : \overline{I} \rightarrow R_1^4$ ), distinct from  $C$ , and a regular map  $\varphi : I \rightarrow \overline{I}$  ( $\overline{s} = \varphi(s)$ ,  $\frac{d\varphi(s)}{ds} \neq 0$  for all  $s \in I$ ) such that the spacelike vectors  $W_1$  of  $C$  and  $\overline{W}_1$  of  $\overline{C}$  are linearly dependent at each pair of corresponding points  $c(s)$  and  $\overline{c}(\overline{s}) = \overline{c}(\varphi(s))$  under  $\varphi$ .

### 3 (1,2)-Bertrand curves in $R_1^4$

Let  $C$  and  $\overline{C}$  be 2-degenerate Cartan curves in  $R_1^4$  and  $\varphi : I \rightarrow \overline{I}$  a regular map ( $\overline{s} = \varphi(s)$ ,  $\frac{d\varphi(s)}{ds} \neq 0$  for all  $s \in I$ ) such that each point  $c(s)$  of  $C$  corresponds to the point  $\overline{c}(\overline{s})$  of  $\overline{C}$  under  $\varphi$  for all  $s \in I$ . Here  $s$  and  $\overline{s}$  are pseudo-arc parameters of  $C$  and  $\overline{C}$  respectively. If the Cartan (1,2)-normal plane at each point  $c(s)$  of  $C$  coincides with the Cartan (1,2)-normal plane at corresponding point  $\overline{c}(\overline{s}) = \overline{c}(\varphi(s))$  of  $\overline{C}$  for all  $s \in I$ , then  $C$  is called the (1,2)-Bertrand curve in  $R_1^4$  and  $\overline{C}$  is called the (1,2)-Bertrand mate of  $C$ .

**Theorem 1** *Let  $C$  be a 2-degenerate Cartan curve in  $R_1^4$  with curvature functions  $k_1, k_2$ . Then  $C$  is a (1,2)-Bertrand curve if and only if there are polynomial*

functions  $\alpha$  and  $\beta$  satisfying

$$\beta(s) \neq 0 \quad (\text{a})$$

$$k_1(s) = 0 \quad (\text{b})$$

$$k_2(s) = \frac{\alpha(s)}{\beta(s)} \quad (\text{c})$$

$$\beta'(s) \neq 0 \quad (\text{d})$$

$$(1 + \alpha'(s))^2 + (\beta'(s))^2 \neq 0 \quad (\text{e})$$

$$\max \deg \{\alpha(s)\} = 1 \quad (\text{f})$$

$$\deg \{\beta(s)\} = 1 \quad (\text{g})$$

for all  $s \in I$ . By (f), we mean the maximum degree of the set containing the polynomial function  $\alpha$  is one.

**Proof.**  $\Rightarrow$ ): Assume that  $C$  is a (1,2)-Bertrand curve, then we can write

$$\bar{c}(\bar{s}) = \bar{c}(\varphi(s)) = c(s) + \alpha(s) W_1(s) + \beta(s) W_2(s) \quad (1)$$

And since the planes spanned by  $\{W_1, W_2\}$  and  $\{\bar{W}_1, \bar{W}_2\}$  coincide, we can also write

$$\bar{W}_1(\bar{s}) = \cos \theta(s) W_1(s) + \sin \theta(s) W_2(s) \quad (2)$$

$$\bar{W}_2(\bar{s}) = -\sin \theta(s) W_1(s) + \cos \theta(s) W_2(s). \quad (3)$$

Notice that  $\sin \theta(s) \neq 0$  for all  $s \in I$ . Because if  $\sin \theta(s) = 0$ , then we get the position  $\bar{W}_1(\bar{s}) = \mp W_1(s)$ . This implies that  $C$  and  $\bar{C}$  coincides. But we know that the Bertrand mate  $\bar{C}$  of  $C$  must be distinct from  $C$ . So, we have  $\sin \theta(s) \neq 0$  for all  $s \in I$ . Now differentiating (1) with respect to  $s$ , we get

$$\begin{aligned} \bar{W}_1(\bar{s}) \frac{d\bar{s}}{ds} &= (1 + \alpha'(s)) W_1(s) + (\alpha(s) - \beta(s) k_2(s)) L(s) \\ &\quad + \beta'(s) W_2(s) + \beta(s) k_1(s) N(s). \end{aligned} \quad (4)$$

Since we assume that the curve  $C$  is (1,2)-Bertrand curve, then the map  $\varphi$  between the pseudo-arc parameters of  $C$  and its Bertrand mate  $\bar{C}$  must be regular. So the following equation holds.

$$\frac{d(\varphi(s))}{ds} = \frac{d\bar{s}}{ds} \neq 0.$$

From (4), we obtain the relations (a), (b) and (c). By the following facts

$$\frac{d\bar{s}}{ds} = \left\langle \bar{W}_1(\bar{s}), \bar{W}_1(\bar{s}) \frac{d\bar{s}}{ds} \right\rangle = (1 + \alpha'(s)) \cos \theta(s) + \beta'(s) \sin \theta(s) \quad (5)$$

$$0 = \left\langle \bar{W}_2(\bar{s}), \bar{W}_1(\bar{s}) \frac{d\bar{s}}{ds} \right\rangle = -(1 + \alpha'(s)) \sin \theta(s) + \beta'(s) \cos \theta(s) \quad (6)$$

we obtain

$$1 + \alpha'(s) = \frac{d\bar{s}}{ds} \cos \theta(s), \quad (7)$$

$$\beta'(s) = \frac{d\bar{s}}{ds} \sin \theta(s). \quad (8)$$

Since  $\frac{d\bar{s}}{ds} \neq 0$  and  $\sin \theta(s) \neq 0$  for all  $s \in I$  in (8), we obtain the relation (d). By using (7) and (8), we get

$$(1 + \alpha'(s))^2 + (\beta'(s))^2 = \left(\frac{d\bar{s}}{ds}\right)^2. \quad (9)$$

Using (9), we obtain (e). The Bertrand mate  $\bar{C}$  of  $C$  is itself a Bertrand curve, therefore the curvature  $\bar{k}_1$  of  $\bar{C}$  is also zero. This means that the components of the curve  $\bar{C}$  consists of polynomials whose third derivative with respect to its pseudo-arc parameter  $\bar{s}$  vanish completely. By using this information and (1), it is obvious that the map  $\varphi$  between the pseudo-arc parameters of  $C$  and  $\bar{C}$  at corresponding points  $c(s)$  and  $\bar{c}(\bar{s})$  respectively, must be linear, that is,  $\frac{d\varphi(s)}{ds} = \frac{d\bar{s}}{ds}$  must be a nonzero constant. Using this fact, (9), and (d) we get the relations (f) and (g).

$\Leftarrow$ ): Now let us think the contrary.

Let  $C$  be a 2-degenerate Cartan curve in  $R_1^4$  with curvature functions  $k_1$  and  $k_2$  and assume that the relations (a),(b),(c),(d),(e),(f) are satisfied for this curve.

Now define a curve  $\bar{C}$  by

$$\bar{c}(s) = c(s) + \alpha(s) W_1(s) + \beta(s) W_2(s) \quad (10)$$

where  $s$  is the pseudo-arc parameter of  $C$ . Differentiating (10) with respect to  $s$ , using the Frenet equations and the hypothesis in the above, we obtain

$$\frac{d\bar{c}(s)}{ds} = (1 + \alpha'(s))W_1(s) + \beta'(s)W_2(s). \quad (11)$$

By using (11), we get

$$\left(\frac{d\bar{c}(s)}{ds}\right)^2 = (1 + \alpha'(s))^2 + (\beta'(s))^2 \neq 0. \quad (12)$$

It is obvious from (12) that  $\bar{C}$  is a regular curve. Let us define a regular map  $\varphi : s \rightarrow \bar{s}$  by

$$\bar{s} = \varphi(s) = \int_0^s \left\langle \frac{d\bar{c}(s)}{ds}, \frac{d\bar{c}(s)}{ds} \right\rangle^{\frac{1}{2}} ds.$$

where  $\bar{s}$  denotes the pseudo-arc parameter of  $\bar{C}$ . Then we obtain

$$\frac{d\bar{s}}{ds} = \frac{d\varphi(s)}{ds} = \sqrt{(1 + \alpha'(s))^2 + (\beta'(s))^2} > 0. \quad (13)$$

Here, notice that

$$\frac{d\bar{s}}{ds} = \lambda \quad (14)$$

is a nonzero constant.

Thus the curve  $\overline{C}$  is rewritten as

$$\overline{c}(\overline{s}) = \overline{c}(\varphi(s)) = c(s) + \alpha(s) W_1(s) + \beta(s) W_2(s). \quad (15)$$

If we differentiate (15) with respect to  $s$ , use the Cartan equations for the 2-degenerate curves in  $R_1^4$  and the hypothesis, we get

$$\lambda \overline{W}_1(\overline{s}) = (1 + \alpha'(s)) W_1(s) + \beta'(s) W_2(s). \quad (16)$$

By using (13), (14) and (16), we can set

$$\overline{W}_1(\overline{s}) = \cos \tau(s) W_1(s) + \sin \tau(s) W_2(s) \quad (17)$$

where

$$\cos \tau(s) = \frac{1 + \alpha'(s)}{\lambda}, \quad (18)$$

$$\sin \tau(s) = \frac{\beta'(s)}{\lambda}. \quad (19)$$

After differentiating (17) with respect to  $s$ , we get

$$\begin{aligned} \lambda \overline{L}(\overline{s}) &= \frac{d \cos \tau(s)}{ds} W_1(s) + \frac{d \sin \tau(s)}{ds} W_2(s) \\ &\quad + (\cos \tau(s) - k_2 \sin \tau(s)) L(s). \end{aligned} \quad (20)$$

Applying the metric  $\langle, \rangle$  on each side of the equation (20), we get

$$(\cos' \tau(s))^2 + (\sin' \tau(s))^2 = 0. \quad (21)$$

From (21), we get

$$\frac{d \cos \tau(s)}{ds} = \frac{d \sin \tau(s)}{ds} = 0.$$

So the  $\tau(s)$  must be the constant function  $\tau_0$ . Thus we obtain

$$\cos \tau_0 = \frac{1 + \alpha'(s)}{\lambda}, \quad (22)$$

$$\sin \tau_0 = \frac{\beta'(s)}{\lambda}. \quad (23)$$

From (17), it holds

$$\overline{W_1}(\overline{s}) = \cos \tau_0 W_1(s) + \sin \tau_0 W_2(s). \quad (24)$$

Now the equation (20) becomes

$$\lambda \overline{L}(\overline{s}) = (\cos \tau_0 - k_2 \sin \tau_0) L(s). \quad (25)$$

Note that, since  $\lambda \neq 0$  in (25), we have

$$\cos \tau_0 - k_2 \sin \tau_0 \neq 0.$$

So we can write the following

$$\overline{N}(\overline{s}) (\cos \tau_0 - k_2 \sin \tau_0) = \lambda N(s). \quad (26)$$

If we differentiate (25) with respect to  $s$ , we get

$$\begin{aligned} \lambda^2 \overline{k_1}(\overline{s}) \overline{W_2}(\overline{s}) &= \frac{d(\cos \tau_0 - k_2 \sin \tau_0)}{ds} L(s) \\ &+ (\cos \tau_0 - k_2 \sin \tau_0) k_1(s) W_2(s). \end{aligned} \quad (27)$$

If we use (b) ( $k_1(s) = 0$ ) in (27), we get

$$\overline{k_1}(\overline{s}) = 0. \quad (28)$$

And therefore the equation (27) reduces to

$$\frac{d(\cos \tau_0 - k_2 \sin \tau_0)}{ds} = 0. \quad (29)$$

Then the nonzero term  $\cos \tau_0 - k_2 \sin \tau_0$  in (29), must be a constant. So we can write

$$\cos \tau_0 - k_2 \sin \tau_0 = \delta \neq 0 \quad (30)$$

where  $\delta$  is a constant.

If we write

$$\frac{\lambda}{\delta} = \ell_0 \neq 0 \quad (31)$$

where  $\ell_0$  is a constant. By using (25), (26) and (31), we get

$$\overline{L}(\overline{s}) = \frac{1}{\ell_0} L(s), \quad (32)$$

$$\overline{N}(\overline{s}) = \ell_0 N(s). \quad (33)$$

Differentiating (33) with respect to  $s$ , we get

$$(\overline{W_1}(\overline{s}) - \overline{k_2}(\overline{s}) \overline{W_2}(\overline{s})) \lambda = \ell_0 (W_1(s) - k_2(s) W_2(s)). \quad (34)$$

From (34), we have

$$(1 + (\overline{k_2}(\overline{s}))^2)\lambda^2 = (\ell_0)^2 (1 + (k_2(s))^2). \quad (35)$$

Using (30) and (31) into (35), we obtain

$$(\overline{k_2}(\overline{s}))^2 = \left[ \frac{\sin \tau_0 + k_2(s) \cos \tau_0}{\cos \tau_0 - k_2(s) \sin \tau_0} \right]^2.$$

Let us take

$$\overline{k_2}(\overline{s}) = \frac{\sin \tau_0 + k_2(s) \cos \tau_0}{\cos \tau_0 - k_2(s) \sin \tau_0}. \quad (36)$$

If we use (17), (30), (31) and (36) into (34), we get

$$\overline{W_2}(\overline{s}) = -\sin \tau_0 W_1(s) + \cos \tau_0 W_2(s). \quad (37)$$

And it is trivial that the Cartan (1,2)-normal plane at each point  $c(s)$  of  $C$  coincides with the Cartan (1,2)-normal plane at corresponding point  $\overline{c}(\overline{s})$  of  $\overline{C}$ . Therefore  $C$  is a (1,2)-Bertrand curve in  $R_1^4$ . ■

## 4 An example of 2-degenerate (1,2)-Bertrand curve in $R_1^4$

Let  $C$  be a curve in  $R_1^4$  defined by

$$c(s) = \left( \frac{s^2}{2}, s, \frac{s^2}{2}, 1 \right).$$

Then we get the Cartan frame and the Cartan curvatures as follows:

$$\begin{aligned} W_1(s) &= (s, 1, s, 1), \\ L(s) &= (1, 0, 1, 0), \\ W_2(s) &= (\sqrt{3}, 0, \sqrt{3}, 1), \\ N(s) &= \left( \frac{s^2}{2} + 2, s, \frac{s^2}{2} + 1, \sqrt{3} \right), \\ k_1(s) &= k_2(s) = 0. \end{aligned}$$

Now we choose the polynomial functions  $\alpha$  and  $\beta$  as follows:

$$\begin{aligned} \alpha(s) &= 0, \\ \beta : R - \{0\} &\rightarrow R; \quad \beta(s) = \sqrt{3}s. \end{aligned}$$

Its Bertrand mate is given by



$$\bar{c}(\bar{s}) = \left( \frac{(\bar{s})^2}{8} + \frac{3\bar{s}}{2}, \frac{\bar{s}}{2}, \frac{(\bar{s})^2}{8} + \frac{3\bar{s}}{2}, 1 + \frac{\sqrt{3}}{2}\bar{s} \right)$$

where  $\bar{s}$  is the pseudo-arc parameter of  $\bar{C}$ , and a regular map  $\varphi : s \rightarrow \bar{s}$  is given by

$$\bar{s} = \varphi(s) = 2s.$$

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